Communication modes in large-aperture approximation

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Received August 24, 2006; accepted November 13, 2006;
posted December 11, 2006 (Doc. ID 74422); published February 15, 2007

Simplified versions of the communication modes in the Fresnel domain are derived when the system apertures are large. The approximate modes, which are in the form of spherical waves and sinc functions with a spherical curvature, give physical insight into the communication modes approach and the basic limits of free-space optical communication systems. They also show that Gabor's information theory is readily derived from the communication modes. © 2007 Optical Society of America

OCIS codes: 050.1940, 050.1960, 060.4510, 070.2580.

In free-space optical communication the number of available spatial channels, and the amount of information that may be transferred, are limited by the geometry of the system and by the wavelength of the optical field. A theory describing this limit was introduced by Gabor in 1961, based on intuitive diffractive optics arguments and sampling theorems.[1] Later, the more systematic theory of the so-called communication modes was developed, based on analytical singular-value decomposition.[2–4] This approach is less intuitive than Gabor's method, but it gives a more complete and accurate description of the information transfer capacity. For example, it can yield the best possible approximation for a given target field distribution,[5] and it also allows for the noise level to be included in the analysis. The communication modes have been found and applied in several different geometries, e.g., communication between two line apertures perpendicular to the optical axis[2,3]; between circular apertures,[2] rectangular apertures, or volumes[4]; and between an axial line and a line[6] or an annular[7] aperture. In all these geometries, the communication modes consist of various modifications of the prolate spheroidal wave functions[2] (PSWFs).

In this Letter we show that in certain usual cases involving sufficiently large apertures it is possible, by use of an appropriate approximation, to find a simpler version of the communication modes.[8] For the geometry consisting of two line or square apertures, perpendicular to the optical axis within the Fresnel diffraction regime, this approximation leads to exactly the same modes as those employed in Gabor's theory.

For simplicity we consider scalar waves in a one-dimensional (1D) geometry as illustrated in Fig. 1. A transmitting aperture of width $2\Delta x_T$ is located in the plane $z=0$, and the field is detected, at a distance z, by a receiving aperture of width $2\Delta x_R$. The Fresnel diffraction integral then describes light propagation at frequency ω according to

$$U_R(x_R) = \int_{-\Delta x_T}^{\Delta x_T} G(x_R,x_T) U_T(x_T) \mathrm{d}x_T, \tag{1}$$

where U_T and U_R are the transmitted and received fields, respectively, and the Green function is

$$G(x_R,x_T) = \frac{\exp(\mathrm{i}kz)}{\sqrt{\mathrm{i}\lambda z}} \exp\left[\mathrm{i}k\frac{(x_R-x_T)^2}{2z}\right], \tag{2}$$

with $k=\omega/c=2\pi/\lambda$ being the wavenumber and λ the wavelength. Let us first introduce new fields across the apertures via the relations

$$a(x_T) = U_T(x_T)\exp(\mathrm{i}kx_T^2/2z), \tag{3}$$

$$b(x_R) = U_R(x_R)\exp(-\mathrm{i}kx_R^2/2z), \tag{4}$$

whereby Eqs. (1) and (2) take on the forms

$$b(x_R) = \int_{-\Delta x_T}^{\Delta x_T} g(x_R,x_T) a(x_T) \mathrm{d}x_T, \tag{5}$$

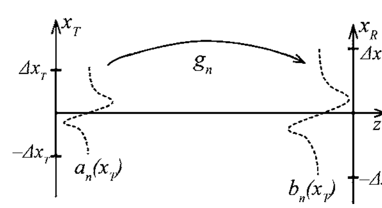

Fig. 1. Illustration of the geometry and notations. The Fresnel domain communication modes $a_n(x_T)$ and $b_n(x_R)$ are normalized PSWFs with quadratic phase factors in both apertures.

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$$g(x_R,x_T)=\frac{\exp(ikz)}{\sqrt{i\lambda z}}\exp\left(-ik\frac{x_Rx_T}{z}\right). \quad (6)$$

These equations simply correspond to the fields from which the characteristic phase curvatures across the apertures have been removed.

In the communication modes approach the Green function appearing in Eqs. (5) and (6) is expanded biorthogonally as[5,9,10]

$$g(x_R,x_T)=\sum_{n=0}^{N} g_n\beta_n(x_R)\alpha_n^*(x_T), \quad (7)$$

where the asterisk denotes the complex conjugate and g_n are the singular values and $\alpha_n(x_T)$ and $\beta_n(x_R)$ the singular functions of Eq. (5), which obey

$$\int_{-\Delta x_T}^{\Delta x_T} g(x_R,x_T)\alpha_n(x_T)\mathrm{d}x_T=g_n\beta_n(x_R), \quad (8)$$

$$\int_{-\Delta x_R}^{\Delta x_R} g^*(x_R,x_T)\beta_n(x_R)\mathrm{d}x_R=g_n^*\alpha_n(x_T), \quad (9)$$

and N is the number of (strongly connected) modes. The mode functions $\alpha_n(x_T)$ and $\beta_n(x_R)$ form complete orthonormal sets of functions in their respective domains. On substitution and integration it follows at once from Eqs. (6), (8), and (9) that[2]

$$|g_n|^2\alpha_n(x_T)=\int_{-\Delta x_T}^{\Delta x_T}\frac{\sin[\Omega_T(x_T-x_T')]}{\pi(x_T-x_T')}\alpha_n(x_T')\mathrm{d}x_T', \quad (10)$$

where $\Omega_T=k\Delta x_R/z$, with a similar, fully analogous integral equation for the modes $\beta_n(x_R)$. It is known that the PSWFs of width Ω_T and scale Δx_T are solutions to Eq. (10),[2] and thus the functions $\alpha_n(x_T)$ are normalized PSWFs. The eigenvalues g_n show a step-like behavior, dropping rapidly to zero after n exceeds the number of degrees of freedom $N=2\Delta x_T\cdot 2\Delta x_R/\lambda z$.

On expanding the field $a(x_T)$ in terms of the transmitting modes as

$$a(x_T)=\sum_n \tilde{a}_n\alpha_n(x_T), \quad (11)$$

where the coefficients are

$$\tilde{a}_n=\int_{-\Delta x_T}^{\Delta x_T} a(x_T)\alpha_n^*(x_T)\mathrm{d}x_T, \quad (12)$$

the field in the transmitting aperture assumes the form

$$U_T(x_T)=\sum_n \tilde{a}_n a_n(x_T)=\sum_n \tilde{a}_n\alpha_n(x_T)\exp(-ikx_T^2/2z), \quad (13)$$

where Eqs. (3) and (11) were used. With the help of Eqs. (4), (8), and (13) the field in the receiving aperture then is

$$U_R(x_R)=\sum_n \tilde{b}_n b_n(x_R)=\sum_n g_n\tilde{a}_n\beta_n(x_R)\exp(ikx_R^2/2z), \quad (14)$$

where $\tilde{b}_n=g_n\tilde{a}_n$. Hence the transmitting and receiving modes $a_n(x_T)$ and $b_n(x_R)$ are converging and diverging PSWFs, respectively.[11] Examples of these modes, together with the coupling coefficient g_n, are schematically shown in Fig. 1.

Let us now consider the limit of a large transmitting aperture, i.e., a situation in which the full width $2\pi/\Omega_T$ of the sinc kernel in Eq. (10) is much narrower than the aperture $2\Delta x_T$. This criterion translates to the condition that the number of modes be large ($N\gg 2$). It is well known that in such a limiting case an integral equation of the type of Eq. (10) admits normalized harmonic exponential functions as eigenfunctions in the integration domain and the eigenvalues are samples from the power spectrum,[12] as can be seen by extending the integration limits from Δx_T to infinity. From Eq. (10) we find the approximate functions $\alpha_{an}(x_T)$ of the transmitting region as

$$\alpha_{an}(x_T)=\frac{1}{\sqrt{2\Delta x_T}}\exp\left(in\frac{\pi x_T}{\Delta x_T}\right), \quad (15)$$

where n is an integer and the approximate eigenvalues g_{an} are

$$g_{an}=1, \quad \text{when } |n|\le|n|_{\max}=2\Delta x_T\Delta x_R/\lambda z, \quad (16)$$

and $g_{an}=0$ otherwise. The integral equation for the modes determines only $|g_{an}|^2$, and we have here chosen g_{an} as real. We emphasize that there is no one-to-one correspondence between the exact (real) PSWFs and the approximate (complex) $\alpha_{an}(x_T)$.[13]

The approximate receiving aperture modes $\beta_{an}(x_R)$ are then obtained with the help of Eq. (8). After straightforward integration the result is

$$\beta_{an}(x_R)=\frac{\exp(ikz)}{\sqrt{i\lambda z}}\sqrt{2\Delta x_T}\,\mathrm{sinc}\left[\frac{k\Delta x_T}{z}\left(x_R-\frac{n\lambda z}{2\Delta x_T}\right)\right], \quad (17)$$

where $\mathrm{sinc}(x)=\sin x/x$. We emphasize that the functions $\beta_{an}(x_R)$ are no longer strictly orthogonal in the domain $(-\Delta x_R,\Delta x_R)$ due to the approximations made, nor do they rigorously solve Eq. (9). The mode $n=0$ is located in the center of the receiving aperture, whereas the modes corresponding to the highest values $n=\pm|n|_{\max}=\pm 2\Delta x_T\Delta x_R/\lambda z$ end up at the aperture edges $x_R=\pm\Delta x_R$.

The full version of the approximate modes $a_{an}(x_T)$ and $b_{an}(x_R)$ in the transmitting and receiving domains additionally contain the associated quadratic phase factors [cf. Eqs. (13) and (14)], i.e.,

$$a_{an}(x_T)=\alpha_{an}(x_T)\exp(-ikx_T^2/2z), \quad (18)$$

$$b_{an}(x_R)=\beta_{an}(x_R)\exp(ikx_R^2/2z). \quad (19)$$

The physical interpretation of these approximate modes is very interesting. The functions $a_{an}(x_T)$ in the transmitting region are converging spherical waves, as shown in Fig. 2. The mode number n determines the tilt of the wavefront, and hence the position of its focus. In the focus, a sinc function described by $\beta_{an}(x_R)$ is created. Each such spot can be regarded as a separate channel for transmitting information. The number of channels is limited by the number of nonzero coupling coefficients, which in this large-aperture approximation according to Eq. (16) is $N=2|n|_{\max}=4\Delta x_T \Delta x_R/\lambda z$, just as for the exact communication modes.

Note that we may view the approximate modes $\alpha_{an}(x_T)$ as basis functions in a discrete, normalized plane-wave (angular spectrum) representation[14] of the field $a(x_T)$ in the transmitting aperture [c.f., Eq. (11)], with the inclination proportional to n. The receiving aperture modes $\beta_{an}(x_R)$ then are shifted sinc functions corresponding to their Fourier transforms (far field).

It is also of fundamental interest to assess the number of separate information channels on the basis of resolution in this approximation, i.e., the spot size of the receiving aperture modes given by Eq. (17). According to the Rayleigh criterion the minimum distance between two resolved spots is $\delta x_R = \lambda z/2\Delta x_T$. Hence if each such image spot is viewed as a separate channel, the receiving aperture $2\Delta x_R$ admits

$$N = \frac{2\Delta x_T \cdot 2\Delta x_R}{\lambda z} \tag{20}$$

independent information channels. This is, again, precisely the number of degrees of freedom of the communication modes.[2]

It is obvious from the formal development of the communication modes that the information system is highly symmetric with regards to the transmitting and receiving domains. Hence the large-aperture approximation could as well be made the other way around: the sinc kernel (with $\Omega_R = k\Delta x_T/z$) for the modes $\beta_n(x_R)$ is assumed to be much narrower than the aperture width $2\Delta x_R$ [c.f., Eq. (10)]. The approximate modes $\beta_{an}(x_R)$ then are normalized, tilted plane waves. Hence the mode functions $a_{an}(x_T)$ in the transmitting region in this approximation would be sinc functions with converging quadratic phase curvature, each of them producing a diverging spherical wave $b_{an}(x_R)$ in the receiving domain. This is precisely the situation presented by Gabor.[1] He expanded the incident 1D field in sinc functions to remove the redundant information according to the Whittaker–Shannon sampling theorem[14] and then counted the number of degrees of freedom as the number of these sinc functions within the finite aperture.

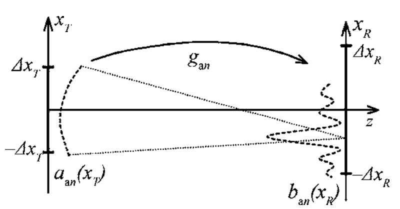

Fig. 2. The approximate communication modes $a_{an}(x_T)$ are converging spherical waves, and they create the modes $b_{an}(x_R)$ that are sinc functions with quadratic phase factors in the receiving domain.

In conclusion, we have developed approximate communication modes in the Fresnel diffraction regime when the apertures are large. These simpler modes lead to interesting physical understanding of the communication mode theory and of the limits imposed on information by free-space optical communications systems. They also show the information theory presented by Gabor can be derived from the exact communication modes.

A. Burvall (previously Thaning) acknowledges a grant (SFI/01/PI.2/B039C) from the Science Foundation of Ireland. P. Martinsson thanks the Swedish Research Council and A. T. Friberg the Swedish Foundation for Strategic Research for financial support. A. Burvall's e-mail address is anna.burvall@nuigalway.ie.

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